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A note on acceleration from product space compactification

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Abstract: We study compactifications of Einstein gravity on product spaces in vacuum and their acceleration phases. Scalar potentials for the dimensionally reduced effective theory are found to be of exponential form and exact solutions are obtained for a class of product spaces. The inflation in our solutions is not sufficient for the early universe. We comment on the possibility of obtaining sufficient inflation by compactification in general.

Keywords: Classical Theories of Gravity, Cosmology of Theories beyond the SM.

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Contents

1. Introduction

In most models of inflation, the source of inflation is a positive potential energy from a scalar field which dominates over the kinetic energy. The precise contribution of the scalar field is still undetermined, except that the potential has to be flat enough in order to produce sufficient inflation. Recently it has been shown that four dimensional accelerating universes can be obtained from pure gravity in higher dimensions [1]–[5]. In these models[1] the role of the inflaton is played by the scalars coming from the higher dimensional metric after dimensional reduction. These models therefore have the attractive feature that the scalar field potential arises in a natural way and is completely determined by gravity.

The no go theorem in ref. [10], states that Einstein gravity coupled to any matter fields with positive kinetic energies can not generate non-singular warped compactifications to de Sitter space dS^d (nor $\mathbb{R}^d$) with finite d dimensional Newton constant, unless there is a positive potential for some scalar fields. An assumption in this no go theorem is that

[1] The solutions of ref. [1] correspond to taking the zero-flux limit of the S-brane solutions in refs. [6, 7]. Previous work on accelerating cosmologies from time dependent flat internal spaces include refs. [8, 9].

the compactified dimensions are time independent. Discarding this assumption, Townsend and Wohlfarth [1] found an accelerating phase for hyperbolic compactifications[2] of the form $\mathbb{R}^{3+1} \times H_n$.

The fact that we need negative curvature for acceleration can be understood from the following observation. For a product space with metric

$$ds^2 = \alpha^2 ds_4^2 + \beta^2 ds_n^2 \,, \tag{1.1}$$

the Hilbert Einstein action is

$$\begin{aligned} S &= \int d^{4+n}x\sqrt{-g}\,\mathcal{R} \\ &= \int d^n x\sqrt{g_n}\int d^4x\sqrt{-g_4}\alpha^4\beta^n\left[\frac{\mathcal{R}_4}{\alpha^2}+\frac{\mathcal{R}_n}{\beta^2}+\cdots\right], \end{aligned} \tag{1.2}$$

where $\mathcal{R}_4$, g_4 ($\mathcal{R}_n$, g_n) are the curvature and metric defined by ds_4^2 (ds_n^2).[3]

When $\beta = \alpha^{-2/n}$, the $4+n$ dimensional action simplifies into the sum of the 4D Hilbert Einstein action plus an action for the scalar field

$$\phi = \ln\beta \,, \tag{1.3}$$

which determines the size of extra dimensions. The line element ds_4^2 is now the metric in 4D Einstein frame.

The second term in (1.2) acts as a scalar field potential for ϕ

$$V = -\mathcal{R}_n e^{-(n+2)\phi} \,. \tag{1.4}$$

Therefore in order to have a positive potential we need to compactify a space with negative curvature. For extra dimensions of constant curvature, that is spherical or hyperbolic spaces, we see from (1.4) that the potential has an exponential form. (For the flat case the potential vanishes.)

Except for very large n, the parameters in the theory are all of order 1 which is a problem in generating inflation from the potential in (1.4). To have large acceleration, we need large V, but this also implies large slope $\partial V/\partial\phi$ and so the kinetic energy quickly catches up according to the equation of motion for ϕ. Therefore the number of e-foldings of such inflationary models is expected to be only of order 1. This is indeed the case for compactification on a single hyperbolic space [1]. If we generalize this setup to extra dimensions which are product spaces for example, this problem might be evaded because

[2]We shall assume that H_n is compact. This can be achieved by taking a quotient of the noncompact hyperbolic space by a discrete subgroup of the isometry group. Since the particular choice of the subgroup and its action does not affect the local Einstein equations, we do not have to specify these details in this paper. Refs. [11, 12] discuss some aspects of compact hyperbolic extra dimensions for cosmology.

[3]The additional terms in the bracket involving derivatives of the scalars α, β are

$$\frac{-6\nabla^2\ln\alpha - 6(\partial\ln\alpha)^2 + (n^2-n)(\partial\ln\beta)^2}{\alpha^2} \,.$$

there are more scalars and more complicated potentials. The situation might be similar to the hybrid model of inflation. We construct a model in section 4 showing that sufficient inflation can be achieved even if all parameters are of order 1.

The plan of the paper is as follows. In section 2, we give a general formulation of Einstein gravity in vacuum for product spaces of flat, spherical and hyperbolic spaces. In section 3, we begin with a review of compactifications on a single hyperbolic space $\mathbb{R}^{3+1} \times H_m$, and then analyze the next simplest case, a product of two compact spaces $\mathbb{R}^{3+1} \times \mathbb{K}_1 \times \mathbb{K}_2$. Although we did not find sufficient inflation for these cases, but we show in section 4 that in principle, similar models with only coefficients of order one can produce eternal inflation. In section 5 we begin analysis of general product spaces $\mathbb{K}_0 \times \mathbb{K}_1 \times \cdots \mathbb{K}_n$, where the large d dimensions does not have to be flat and d does not have to be 3. We have found exact solutions for the case $\mathbb{K}_1 = \mathbb{K}_2 = \cdots = \mathbb{K}_n$, where $\mathbb{K}_i$ can be a spherical, hyperbolic or flat space. Finally we comment on our results in section 6.

2. Generic case

In this section we consider Einstein gravity in vacuum for spacetimes which are products of an arbitrary number of spaces, which are each either flat, spherical or hyperbolic. In subsection 2.1 we first give the vacuum Einstein equations, and then treating the scale factors as scalar fields we rewrite the theory from the dimensionally reduced viewpoint in subsection 2.2. The advantage of this approach is that we can now apply our knowledge of scalar field inflation.

2.1 Einstein equations for product spaces in vacuum

Let us consider spacetimes of the form

$$\mathbb{R} \times \mathbb{K}_0 \times \mathbb{K}_1 \times \cdots \times \mathbb{K}_n \, , \tag{2.1}$$

which are products of flat, spherical and hyperbolic spaces in addition to the time direction. Each factor of $\mathbb{K}_i$ can be either flat, spherical or hyperbolic which we label by a numbers ϵ_i, which are $0, 1$ or -1, respectively. The dimension of each space $\mathbb{K}_i$ will be denoted m_i for $i = 0, \ldots, n$ so the total dimension is $\sum_{i=0}^{n} m_i + 1$. We define m_0 by d, and we will find the dimensional reduction of the higher dimensions to $\mathbb{K}_0$.

Our metric ansatz for the vacuum solution is

$$ds^2 = -\mathrm{e}^{2A(t)} dt^2 + \sum_{i=0}^{n} \mathrm{e}^{2B_i(t)} ds_i^2 \, , \tag{2.2}$$

where $ds_i^2 = \bar{g}_{ab}^{(i)} dx^a dx^b$ is the metric for the space $\mathbb{K}_{(m_i, \epsilon_i)}$ given by

$$ds_i^2 = \begin{cases} d\chi^2 + \chi^2 d\Omega_{m_i-1}^2 \, , & \text{for} \quad \epsilon_i = 0 \, , \\ d\chi^2 + \sin^2\chi \, d\Omega_{m_i-1}^2 \, , & \text{for} \quad \epsilon_i = 1 \, , \\ d\chi^2 + \sinh^2\chi \, d\Omega_{m_i-1}^2 \, , & \text{for} \quad \epsilon_i = -1 \, . \end{cases} \tag{2.3}$$

For each component of the product space, the Ricci tensor is given by

$$\bar{R}^{(i)}_{ab} = \epsilon_i (m_i - 1) \bar{g}^{(i)}_{ab} \tag{2.4}$$

so the Ricci tensor for the full metric (2.2) is relatively simple

$$\begin{aligned} R_{tt} &= -\sum_{i=0}^{n} m_i \left(\ddot{B}_i + \dot{B}_i^2 - \dot{A}\dot{B}_i \right) , \\ R^{(i)}_{ab} &= \left\{ \mathrm{e}^{2B_i - 2A} \left[\ddot{B}_i + \dot{B}_i \left(-\dot{A} + \sum_{j=0}^{n} m_j \dot{B}_j \right) \right] + \epsilon_i (m_i - 1) \right\} \bar{g}^{(i)}_{ab} \, . \end{aligned} \tag{2.5}$$

From the above expressions, we see that the equations simplify using the gauge condition

$$-A + \sum_{j=0}^{n} m_j B_j = 0 \, . \tag{2.6}$$

This is merely a gauge condition corresponding to a time reparametrization since all the functions depend only on time. In this gauge, the vacuum Einstein equations reduce to

$$\sum_{i=0}^{n} m_i \left(\ddot{B}_i + \dot{B}_i^2 - \dot{A}\dot{B}_i \right) = 0 \, , \tag{2.7}$$

$$\ddot{B}_i + \epsilon_i (m_i - 1) \mathrm{e}^{2A - 2B_i} = 0 \, . \tag{2.8}$$

Although the general solutions are not easy to obtain due to the coupling between the differential equations, we have obtained exact solutions for some particular cases. We will discuss some of these solutions below.

2.2 Effective theory in lower dimension

Let us assume that $\mathbb{K}_0$ corresponds to the large spatial dimensions in which we live, so the $\mathbb{K}_i$'s are extra dimensions. We would like to dimensionally reduce the theory to $\mathbb{K}_0$ and obtain an effective theory of d dimensional gravity coupled to scalar fields.

As noted in section 1, the metric of the d dimensional Einstein theory is different from the d dimensional part of the full metric. We rewrite (2.2) as

$$ds^2 = \alpha^2 a^2 \left(-d\eta^2 + ds_0^2 \right) + \sum_{i=1}^{n} a_i^2 ds_i^2 \, , \tag{2.9}$$

where η is the conformal time in the d dimensional Einstein frame, a is the d dimensional scale factor and

$$\alpha = \prod_{i=1}^{n} \alpha_i \, , \qquad \text{with} \qquad \alpha_i = a_i^{-\frac{m_i}{d-1}} \, . \tag{2.10}$$

The Einstein tensor is

$$\begin{aligned} G_{00} &= \frac{d(d-1)}{2} \left(\frac{a'}{a} \right)^2 - \rho \, a^2 \, , \\ G_{11} &= -\frac{(d-1)(d-4)}{2} \left(\frac{a'}{a} \right)^2 - (d-1) \frac{a''}{a} - p \, a^2 \, , \end{aligned} \tag{2.11}$$

where ρ and p can be identified with the energy density and pressure in d dimensional space. The other non-vanishing components of the Einstein tensor give redundant equations. In the above we have denoted the derivative of η by a prime: $f' = df/d\eta$.

From ρ and p we deduce the kinetic and potential terms[4] for the scalar fields

$$\begin{aligned} K &= \frac{\rho+p}{2} = \sum_{i=1}^{n} \frac{m_i(m_i+d-1)}{2(d-1)a^2}\phi_i'^2 + \sum_{i>j=1}^{n} \frac{m_i m_j}{(d-1)a^2}\phi_i'\phi_j' - \epsilon_0 \frac{d-1}{2a^2}\,, \\ &= \sum_{i=1}^{n} \frac{1}{2a^2}\varphi_i'^2 + \sum_{i>j=1}^{n} \frac{2c_i c_j}{a^2}\varphi_i'\varphi_j' - \epsilon_0 \frac{d-1}{2a^2}\,, \end{aligned} \tag{2.12}$$

$$\begin{aligned} V &= \frac{\rho-p}{2} = \sum_{i=1}^{n}(-\epsilon_i)\frac{m_i(m_i-1)}{2} e^{-\frac{2}{d-1}\left((m_i+d-1)\phi_i + \sum_{j\neq i}^{1\leq j\leq n} m_j\phi_j\right)} - \epsilon_0 \frac{(d-1)^2}{2a^2}\,, \\ &= \sum_{i=1}^{n}(-\epsilon_i)\frac{m_i(m_i-1)}{2} e^{\sqrt{\frac{2}{d-1}}\left(\varphi_i/c_i + 2\sum_{j\neq i}^{1\leq j\leq n} c_j\varphi_j\right)} - \epsilon_0 \frac{(d-1)^2}{2a^2}\,, \end{aligned} \tag{2.13}$$

where

$$c_i = \sqrt{\frac{m_i}{2(m_i+d-1)}}\,, \tag{2.14}$$

and the scalar fields ϕ_i, φ_i are defined by

$$a_i = e^{\phi_i}, \qquad \phi_i = -\frac{\sqrt{2(d-1)}\,c_i}{m_i}\varphi_i\,. \tag{2.15}$$

From (2.10) we then have

$$\alpha_i = e^{\sqrt{\frac{2}{d-1}}\,c_i\varphi_i}\,. \tag{2.16}$$

The last terms in (2.12) and (2.13) are the contributions from the curvature of the d dimensional space.

The effective lagrangian is $(8\pi G_{d+1} = 1)$ [5]

$$L = \sqrt{-g}\left(\frac{\mathcal{R}}{2} + K - V\right) = a^{d+1}\left[-\frac{d(d-1)}{2}\left(\frac{a'}{a^2}\right)^2 + K - V\right]. \tag{2.17}$$

The action for the scalar fields can also be derived from the Einstein-Hilbert action in higher dimensions, up to total derivatives, as we explain in appendix A.

It is well known that off-diagonal terms omitted in (2.9) are gauge fields from the lower dimensional viewpoint. We can include their effect by adding Yang-Mills lagrangian to the effective action, but the gauge coupling will be a function of the scalar fields. If we redefine the gauge field to absorb these factors so that the gauge coupling is constant, we are effectively introducing charges to the scalar fields. We will not pursue this effect here.

3. Simple product spaces

In this section we study the acceleration of the product spaces $\mathbb{R}^{3+1} \times H_m$ and $\mathbb{R}^{3+1} \times \mathbb{K}_1 \times \mathbb{K}_2$.

[4] Potentials coming from compactifications were discussed in ref. [13].

[5] In our convention, the factor $\sqrt{-g_4} = a^{d+1}$ is included in L, and so the action is just $S = \int d^{d+1}x L$.

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3.1 $\mathbb{R}^{d+1} \times H_m$

Our ansatz for the spacetime metric of $\mathbb{R}^{d+1} \times \mathbb{K}$ is

$$ds^2 = \alpha^2 \left(- a^{2d}\, dt^2 + a^2\, dx_{(d)}^2 \right) + \alpha^{-\,2(d-1)/m} r_h^2\, ds_1^2 \,, \tag{3.1}$$

and $\mathbb{K}$ is taken to be an m-dimensional compact hyperbolic space with the metric ds_1^2 given by (2.3) for $\epsilon_1 = -1$. The ansatz (3.1), with

$$\alpha^2(t) = \left(\frac{e^{\,d\lambda_0 t}}{K(t)} \right)^{2m/(m-1)(d-1)} \,, \qquad a^2(t) = \left(\frac{K(t)}{e^{[(m+d-1)/m]\lambda_0 t}} \right)^{2m/(m-1)(d-1)} \,, \tag{3.2}$$

solves the vacuum Einstein equations for

$$K(t) = \frac{\lambda_0 r_h}{(m-1)}\, \frac{\beta}{\sinh\left[\lambda_0 \beta |t - t_1|\right]} \,, \qquad \beta = \sqrt{\frac{d(m+d-1)}{m}} \,, \tag{3.3}$$

where t_1 is an integration constant. This solution was first found in ref. [1] when $d = 3$. Henceforth we take $\lambda_0 = 1$, $r_h = 1$, and shift the time so that $t_1 = 0$.

The proper time for a four-dimensional observer (i.e., $d = 3$) is measured by τ, with $d\tau = a^3(t)\, dt$. The metric (3.1) then takes the form

$$ds^2 = \alpha^2(\tau)(-d\tau^2 + a^2(\tau)\, dx^2) + \alpha^{-\,4/m}(\tau) ds_1^2 \,. \tag{3.4}$$

According to (2.13), the potential of φ is

$$V = \frac{m(m-1)}{2}\, e^{\varphi/c_m} \,, \tag{3.5}$$

where

$$c_m = \sqrt{\frac{m}{2(m+2)}} \,. \tag{3.6}$$

In order to have sufficient inflation, the potential can not be too steep, otherwise φ rolls down the hill too quickly and the inflationary era will be too short. The slow roll condition $V^{-1} \frac{dV}{d\varphi} \ll 1$ is satisfied if $c_m \gg 1$. However, c_m is bounded by $1/2 \le c_m < \sqrt{1/2}$.

The $4D$ spacetime is expanding if

$$\frac{da}{d\tau} > 0 \quad \Longrightarrow \quad n_1(t) \equiv 1 + \frac{m\beta}{m+2}\, \coth(\beta\, t) < 0 \,, \tag{3.7}$$

with β defined in (3.3). The expansion is accelerating only if

$$\frac{d^2 a}{d\tau^2} > 0 \quad \Longrightarrow \quad n_2(t) \equiv \frac{m(m-1)\beta^2}{(m+2)^2 \sinh^2 [\beta\, t]} > (n_1(t))^2 \,. \tag{3.8}$$

The conditions (3.7) and (3.8) are satisfied simultaneously for $t < 0$ in a certain interval. A small but negative time t actually corresponds to a positive proper time τ. Specifically, in the limit $0 < t \ll 1$, we find

$$\tau \sim - \left(\frac{1}{t} \right)^{(m+2)/2(m-1)} \,, \qquad a \sim \left(\frac{1}{t} \right)^{m/2(m-1)} \,. \tag{3.9}$$

n	$t_i \cong$	$t_f \cong$	Ratio $f = \frac{a(\tau_2)}{a(\tau_1)}$
2	-0.7367	-0.1991	1.99
3	-0.7249	-0.1359	2.25
4	-0.7259	-0.1051	2.48
5	-0.7287	-0.0861	2.68
6	-0.7316	-0.0731	2.83
7	-0.7341	-0.0636	3.04

Table 1: The period of the accelerated expansion for $d = 3$.

This also implies that the singularity of the function $K(t)$ at $t = 0_+$ corresponds to $\tau = -\infty$ in Einstein frame.

An intuitive understanding of why the acceleration occurs at negative time t is given in [1, 5]. Numerical studies show that the number of e-foldings[6]

$$N = \ln\left(\frac{a(t_f)}{a(t_i)}\right) = \int_{t_i}^{t_f} dtH \tag{3.10}$$

is of order 1, see table 1. (Here $H = a^{-1}(da/d\tau)$ is the Hubble parameter.) The solution for the spacetime metric $\mathbb{R}^{3+1} \times H_m$ is therefore not applicable for implementing inflation in the early universe. We note that the initial condition $\varphi'(t_i)$ does not significantly affect the value of N, because a larger initial velocity will shoot up the hill to a higher point, which is good for inflation, but it will roll down faster as the slope is also larger at higher points.

3.2 $\mathbb{R}^{3+1} \times \mathbb{K}_1 \times \mathbb{K}_2$

In this section we consider the spacetimes $\mathbb{R}^{3+1} \times \mathbb{K}_1 \times \mathbb{K}_2$, which are the product of a large 3 dimensional flat space and two compact spaces. The ansatz for the spacetime metric is

$$\begin{aligned} ds^2 &= \alpha^2(\eta)a^2(\eta)(-d\eta^2 + dx^2) + \beta_1^2(\eta)ds^2_{\mathbb{K}_1} + \beta_2^2(\eta)ds^2_{\mathbb{K}_2}\,, \\ \alpha &= \alpha_1\alpha_2\,, \qquad \alpha_1 = e^{c_m\varphi_1}\,, \qquad \alpha_2 = e^{c_n\varphi_2}\,, \\ \beta_1 &= \alpha_1^{-2/m}\,, \qquad \beta_2 = \alpha_2^{-2/n}\,, \end{aligned} \tag{3.11}$$

where the metric for $\mathbb{K}_1$ and $\mathbb{K}_2$ are of the form given in (2.3), and their dimensions are m and n.

As shown in section 2, the Einstein theory dimensionally reduced to 4D is equivalent to 4D Einstein gravity coupled to scalar fields φ_1, φ_2 with the kinetic term

$$K = \frac{1}{a^2}\left(\frac{1}{2}\left(\varphi_1'^2 + \varphi_2'^2\right) + 2c_m c_n \varphi'_1\varphi'_2\right), \tag{3.12}$$

and the potential

$$V = V_{m,\,\epsilon_1}e^{2c_n\varphi_2} + V_{n,\,\epsilon_2}e^{2c_m\varphi_1}\,, \tag{3.13}$$

[6]The initial time t_i and final time t_f are defined by the times when the scale factor a starts and stops to accelerate, respectively.

where $V_{m,\,\epsilon_i}$ are given by

$$V_{m,\,\epsilon_i} = -\,\epsilon_i\,\frac{m(m-1)}{2}\,e^{\varphi_i/c_m}\,. \tag{3.14}$$

Let us define the new scalars

$$\psi_1 = \varphi_1 + 2c_m c_n \varphi_2\,, \qquad \psi_2 = c_{mn}\varphi_2\,, \tag{3.15}$$

so that the kinetic terms take the canonical form $K = \frac{1}{2a^2}(\psi'^2_1 + \psi'^2_2)$. In these variables the potential term becomes

$$\begin{aligned} V &= -\epsilon_1 \frac{m(m-1)}{2}\,e^{\psi_1/c_m} - \epsilon_2 \frac{n(n-1)}{2}\,e^{2c_m\psi_1 + c_{mn}\psi_2/c_n}\,, \\ c_{mn} &= \sqrt{\frac{2(m+n+2)}{(m+2)(n+2)}}\,. \end{aligned} \tag{3.16}$$

This resembles a hybrid model for inflation.

4. Parameters of order 1 vs. 60 e-foldings

For the spacetimes $\mathbb{R}^{3+1} \times \mathbb{K}_1 \times \mathbb{K}_2$ studied in section 3.2, the scalar potential is

$$V = e^{2\psi_1} + e^{\psi_1 + \sqrt{3}\psi_2}\,, \tag{4.1}$$

for $m = n = 2$, $\epsilon_1 = \epsilon_2 = -1$. One might be tempted to claim that it is impossible for this potential to generate sufficient inflation because all the parameters here are of order one. How can we obtain 60 e-foldings from such a potential?

Although it is true that for this particular model we can not obtain sufficient inflation, the argument above is over-simplifying the problem. Let us demonstrate this with an explicit example showing that the interaction between two fields can lead to a surprising difference from the naive expectation.

4.1 Numerical results

The potential we examine

$$V = e^{2\psi_1/3} + e^{3\psi_1 + \sqrt{3}\psi_2}\,, \tag{4.2}$$

is of the same form as the one in (4.1), where all parameters are of order one. The only difference between these two potentials is the coefficients of ψ_1 in the exponents.

The equations of motion for the scalar fields expressed in the physical time t ($dt = a d\eta$) are

$$\ddot{\psi}_i + 3H\dot{\psi}_i + \frac{\partial V}{\partial \psi_i} = 0\,, \tag{4.3}$$

and the Friedman equation

$$H^2 = \frac{1}{3}(K + V) \tag{4.4}$$

determines the evolution of the scale factor $a(t)$.

We have studied the expansion of this model numerically. For two 2nd order differential equation, we can set four initial values $\psi_1, \psi_2, \dot{\psi}_1$ and $\dot{\psi}_2$. If we fix the initial condition

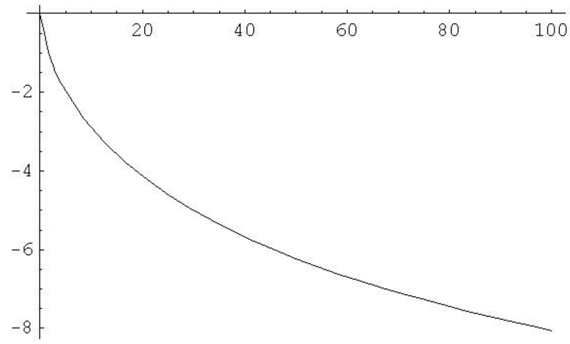

Figure 1: The evolution of ψ_1 for the potential (4.2).

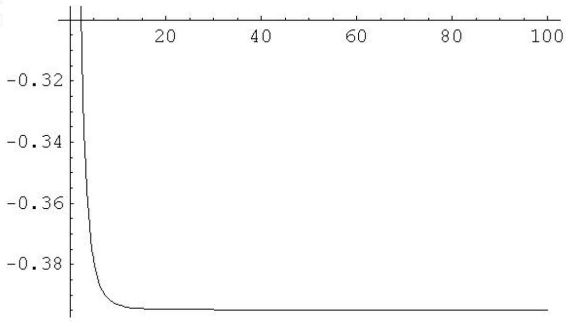

Figure 2: The evolution of ψ_2 for the potential (4.2).

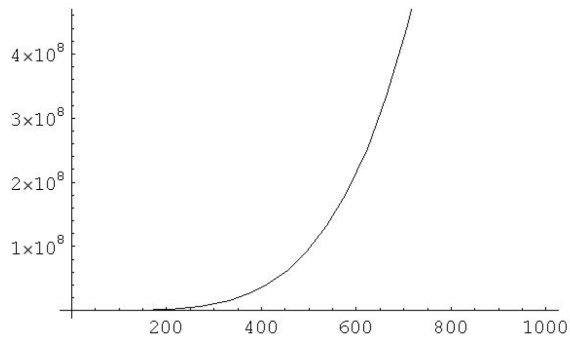

Figure 3: Evolution of the scale factor $a(t)$ for the potential (4.2). The acceleration never stops.

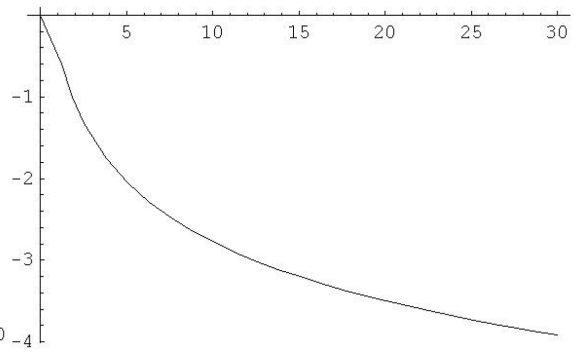

Figure 4: The evolution of ψ_1 for the potential (4.1).

so that it coincides with the starting point of acceleration, we first set three of the initial values, $\psi_1(0)$, $\dot{\psi}_1(0)$, $\dot{\psi}_2(0)$, and this fixes the initial value of $\psi_2(0)$ by requiring that $\ddot{a} = 0$, or equivalently $V = 2K$ at $t = 0$. A set of such initial conditions is $\psi_1(0) = 0, \psi_2(0) = 0, \dot{\psi}_1 = 1, \dot{\psi}_2 = 1$. Note that all the initial values of this set are of order 1, and the results will be qualitatively the same for variations of order 1 for the initial conditions.

Our numerical study showed that the acceleration never stops, but asymptotes to zero acceleration, $\ddot{a} \to 0$. The evolution of ψ_1, ψ_2 and the scale factor $a(t)$ are given in figures 1, 2 and 3, respectively. In contrast, the results for the potential (4.1) from dimensional reduction are plotted in figures 4–6. Note that the time scales of the plots are quite different.

The Hubble parameter $H = \dot{a}/a$ is found to be $0.004, 0.0004, 0.00004$ at $T = 10^3$, 10^4, 10^5, respectively. The corresponding e-folding number is $21, 32, 42$, respectively. It is remarkable that while the two potentials (4.1) and (4.2) are so similar, their spacetime evolutions are so drastically different.

The behavior of the coupled system is quite interesting. We see from the figure of ψ_2 that it asymptotes to a constant as $t \to \infty$. The only difference we noticed between the potentials (4.1) and (4.2) which might be crucial here is that the relative magnitude of

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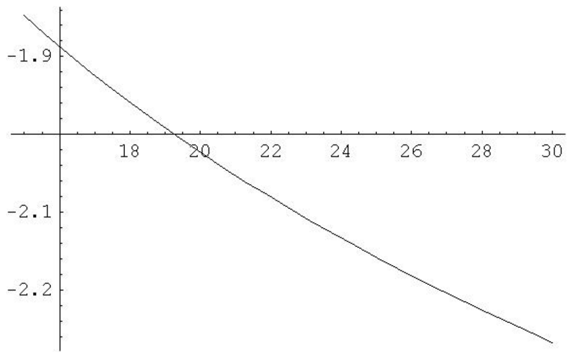

Figure 5: The evolution of ψ_2 for the potential (4.1).

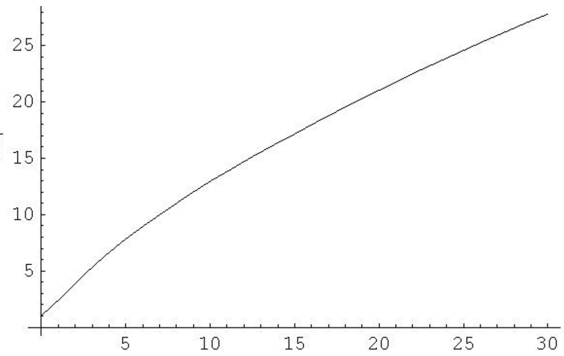

Figure 6: Evolution of the scale factor $a(t)$ for the potential (4.1). Acceleration for a very short period of order 1 occurs near the origin $t = 0$.

the coefficients of ψ_1 in the two exponents in V is reversed. It will be very interesting to understand better the mechanism behind the model (4.2) that leads to eternal inflation. This knowledge will be important for inflationary model building.

4.2 Exact solutions

In the previous subsection we studied numerically models of two scalar fields with the exponential potential

$$V = e^{b\psi_1} + e^{c\psi_1 + d\psi_2}\,. \tag{4.5}$$

In fact, exact solutions can also be obtained. From the ansatz

$$\psi_i = p_i \ln \tau + q_i\,, \qquad H = \frac{h}{\tau}\,, \tag{4.6}$$

we find the solution

$$\begin{aligned} p_1 &= -\frac{2}{b}\,, & p_2 &= -\frac{2(b-c)}{bd}\,, \\ e^{bq_1} &= \frac{2(3h-1)(d^2 - c(b-c))}{b^2 d^2}\,, & e^{cq_1+dq_2} &= \frac{2(3h-1)(b-c)}{bd^2}\,, \end{aligned} \tag{4.7}$$

and

$$h = \frac{2(d^2 + (b-c)^2)}{b^2 d^2}\,. \tag{4.8}$$

These special solutions are either acceleration for all τ or deceleration for all τ. The condition for acceleration is $h > 1$. For the potentials from compactification on $\mathbb{K}_1 \times \mathbb{K}_2$ in section 3.2,

$$h = \frac{m_1 + m_2}{m_1 + m_2 + 2} < 1\,, \tag{4.9}$$

and there is no inflation. For the hypothetical model (4.2) we studied numerically in the previous subsection, $h = 38/3 > 1$.

To understand the form and simplicity of these solutions it is useful to change variables from ψ to φ. For the case where the extra dimensions are $H_2 \times H_2$, we have

$$\begin{aligned} b &= 2\,, \qquad c = 1\,, \qquad d = \sqrt{3}\,, \\ p_1 &= -1\,, \qquad p_2 = -\frac{1}{\sqrt{3}}\,. \end{aligned} \tag{4.10}$$

The time dependence of the φ's is given by

$$\varphi_1 = -\frac{2}{3}\ln\tau + \text{constant}\,, \qquad \varphi_2 = -\frac{2}{3}\ln\tau + \text{constant}\,, \tag{4.11}$$

so we find φ_1 and φ_2 have the same time dependence. This result is consistent with the fact that $\varphi_1 = \varphi_2$ is a symmetry axis of the potential; our numerical simulations confirm the intuition that most solutions tend to fall along this line of attraction due to energy lost to cosmic friction.

Comparing these solutions to the general solution for a single product space H_m, we see that these exact solutions can be understood as the large τ limit of the generic solutions [5]. If we understand these solutions as representing future infinity, we see that although we can start with generic solutions, their eventual fate seems to be that the different hyperbolic spaces attain the same time dependence.

In the next section we will find exact solutions where the product spaces all have the same time dependence and we find that inflation is not improved. If all solutions are generically attracted to the symmetry axis, then eternal inflation is apparently ruled out when the product spaces are all of the same type.

Apparently the ansatz (4.6) can be easily applied to an arbitrary number of scalar fields with exponential potential. Exact solutions of this class will be reported in a follow up paper.

5. Product spaces

The eternal inflation in the hybrid model of the previous section motivates us to look for solutions of more general product spaces as extra dimensions.

5.1 Equal product spaces

In this subsection, we discuss the case of n copies of $\mathbb{K}_i$: $\epsilon_0 = 0$, $m_0 = d$, $m_1 = \cdots = m_n = m$, $B_1 = \cdots = B_n = B$ and $\epsilon_1 = \cdots = \epsilon_n = \epsilon$. For this case the equation for B_0 can be solved

$$B_0 = \alpha_0 t + \beta_0\,, \tag{5.1}$$

where α_0 and β_0 are integration constants, and the variable

$$A = d(\alpha_0 t + \beta_0) + mnB \tag{5.2}$$

is determined by the gauge condition (2.6). Then the only equations we need to solve are (2.7) and (2.8) for B

$$\alpha_0^2 d(1-d) + mn[\ddot{B} + (1-mn)\dot{B}^2 - 2\alpha_0 d\dot{B}] = 0\,, \tag{5.3}$$

$$\ddot{B} + \epsilon(m-1)\mathrm{e}^{2d(\alpha_0 t+\beta_0)+2(mn-1)B} = 0\,. \tag{5.4}$$

The change of variables

$$B = \sqrt{\frac{m-1}{mn-1}} f - \frac{d}{mn-1}(\alpha_0 t + \beta_0)\,,$$
$$A = mn\sqrt{\frac{m-1}{mn-1}} f - \frac{d}{mn-1}(\alpha_0 t + \beta_0)\,. \tag{5.5}$$

simplifies the equations of motion

$$\ddot{f} + \epsilon\lambda \mathrm{e}^{2\lambda f} = 0\,, \qquad \leftrightarrow \qquad \dot{f}^2 + \epsilon \mathrm{e}^{2\lambda f} = f_0^2\,, \tag{5.6}$$

where $\lambda = \sqrt{(m-1)(mn-1)}$. The solutions of this differential equation are

$$f(t) = \begin{cases} \frac{1}{\lambda}\ln\left(\frac{f_0}{\sinh[\lambda f_0|(t-t_1)|]}\right) & \epsilon = -1. \\ f_0(t-t_1) & \epsilon = 0. \\ \frac{1}{\lambda}\ln\left(\frac{f_0}{\cosh[\lambda f_0(t-t_1)]}\right), & \epsilon = +1\,, \end{cases} \tag{5.7}$$

where the value of f_0 is fixed from (5.3) to be

$$f_0 = \sqrt{\frac{(mn+d-1)d}{mn(m-1)(mn-1)}}\,\alpha_0\,. \tag{5.8}$$

The constants α_0 and β_0 can be set as $\alpha_0 = 1$ and $\beta_0 = 0$ by rescaling and shifting time t.

Following the ansatz (2.2), the $(d{+}1)$-dimensional Einstein frame metric for this generic case is

$$ds_{d+1}^2 = \mathrm{e}^{\frac{2mn}{d-1}B}\left(-\mathrm{e}^{2A}dt^2 + \mathrm{e}^{2B_0}dx_d^2\right). \tag{5.9}$$

When $\mathbb{K}$ is a hyperbolic space (i.e. $\epsilon = -1$), we have

$$ds_{d+1}^2 = -a^{2d}dt^2 + a^2 dx^2\,, \tag{5.10}$$

with

$$a(t) = h_1(t)h_2(t)\,, \quad h_1(t) = e^{-\frac{mn+d-1}{(d-1)(mn-1)}t}\,, \quad h_2(t) = \left(\frac{f_0}{\sinh[\lambda f_0|(t-t_1)|]}\right)^{\frac{mn}{(d-1)(mn-1)}}\,. \tag{5.11}$$

In terms of the proper time coordinate τ, with

$$d\tau = a^d dt\,, \tag{5.12}$$

the metric (5.10) can be written in the standard FLRW form as

$$ds_{d+1}^2 = -d\tau^2 + a^2(\tau)dx_d^2\,. \tag{5.13}$$

The accelerating phase occurs when

$$\partial_\tau a > 0\,, \qquad \partial_\tau\partial_\tau a > 0\,, \tag{5.14}$$

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which, after a straightforward calculation, give

$$-\sqrt{mnd}\coth\theta - \sqrt{mn+d-1} > 0\,,$$
$$d(mn-1)\mathrm{csch}^2\theta - (\sqrt{mnd}\coth\theta + \sqrt{mn+d-1})^2 > 0\,, \tag{5.15}$$

where $\theta = \lambda f_0(t-t_1)$. It is interesting that this equation is dependent only on the product mn and not m or n separately. For these particular solutions we find that the period of acceleration and the number of e-foldings are the same for H_{mn} and product spaces such as n copies of H_m.

Analysis of these conditions shows that acceleration phases exist for all values of $d \geq 1, m \geq 2, n \geq 1$. We find larger e-foldings for smaller d and larger mn. For $d = 1$, inflation can last forever, although the acceleration eventually approaches to zero. For $d = 3$, the number of e-foldings is of order one for mn of order 100 or smaller.

We finally note that for the case of a simple product space, going to higher dimensions d and m actually decreases the amount of inflation. In arbitrary d dimensions, the slow roll condition becomes

$$\left(\frac{V'}{V}\right)^2 \ll \frac{d}{(d-1)^2} \tag{5.16}$$

which is consistent with our results that inflation for $d = 1$ can last forever. For the spacetime $R^{1,d} \times H_m$ the potential is of exponential form and we find

$$\left(\frac{V'}{V}\right)^2 = 4\,\frac{m+d-1}{m(d-1)} \tag{5.17}$$

which is of order $1/d$ for the case where we take $m \approx d$, and also for the case where we take m large. While going to large d does flatten out the potential which is good for obtaining large inflation, going to large d also increases the cosmic friction. Going to large d therefore does not help us satisfy the slow roll conditions and does not increase the amount of inflation.

Apparently there is no room for a realistic inflation model for the early universe using the solutions found in this subsection.

6. Discussion

In this paper, we generalized the work of [1] and considered compactifications of product spaces of flat, spherical and hyperbolic spaces. Unfortunately, for all the cases we studied, the acceleration phases are still not sufficient. Nevertheless, we can not yet conclude that 60 e-foldings cannot be achieved from pure gravity compactification.

For the product spaces considered in section 5, the exact solutions we found are only very special solutions. In general each space in the product can have a different scale factor, but we have only found solutions with a single overall scale factor for the extra dimensions. At least numerically, one can also study product spaces for which we did not obtain exact solutions. In view of the importance of interactions between scalar fields which we discussed in section 4, the phase diagram for the solutions can be very complicated and the implication of the special solutions can not be taken too far.

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Since we know that radiation induces deceleration, adding off-diagonal components into the full spacetime metric does not seem so promising. Similarly, in view of the no go theorem of [10], adding matter fields will probably not help. It would be interesting to know whether modifications to Einstein theory, such as Brans Dicke theory or adding Gauss Bonnet terms will admit solutions with sufficient acceleration.

If we assume that the compactified space is stabilized before nucleosynthesis in order not to violate observational data, it will be hard to apply the acceleration associated with the shrinking of extra dimensions to explain the acceleration of the present universe. This is why we have kept in mind the inflation of the early universe as our main application. But a more careful study is needed before this possibility is completely ruled out.

Although this original setup of Townsend and Wohlfarth does not provide large e-foldings for a three dimensional universe, it is possible that with modification, solutions with large amounts of inflation can be found from a suitable high energy starting point using gravity as the source of inflation. In a follow up paper we will discuss the details of these more complicated scenarios.

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A. Dimensional reduction

Consider a product space metric

$$ds^2 = G_{MN}dx^M dx^N = e^{2\phi} g_{\mu\nu} dx^\mu dx^\nu + \sum_{i=1}^{n} e^{2\phi_i} ds_i^2\,, \qquad \mu = 0, \ldots, d\,, \tag{A.1}$$

where each ds_i^2 characterized by ϵ_i can be flat ($\epsilon_i = 0$), spherical ($\epsilon_i = 1$) or hyperbolic space ($\epsilon_i = -1$) with dimensions m_i. The total spacetime dimension is $D = d+1+\sum_{i=1}^{n} m_i$. For this ansatz, the D-dimensional Hilbert-Einstein action decomposes as

$$\begin{aligned} S &= \int d^D x \sqrt{-G}\,\mathcal{R}_G\,, \\ &= \int d^m x \int d^{d+1} x \sqrt{-g}\,\exp\left[(d-1)\phi + \sum_{i=1}^{n} m_i\phi_i\right] \times \\ &\quad \times \left\{ \mathcal{R}_g - 2d\nabla^2\phi - d(d-1)(\partial\phi)^2 + \sum_{i,j=1}^{n} m_i m_j\, g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_j - \sum_{i=1}^{n} m_i(\partial\phi_i)^2 + \right. \\ &\quad \left. + \sum_{i=1}^{n} \epsilon_i m_i(m_i-1)e^{2\phi-2\phi_i} \right\}. \end{aligned} \tag{A.2}$$

If we go to Einstein frame using

$$(d-1)\phi + \sum_{i=1}^{n} m_i \phi_i = 0\,, \tag{A.3}$$

then the term $-2d\sqrt{-g}\,\nabla^2\phi$ becomes a total derivative which can be neglected. Finally, the d-dimensional effective action becomes

$$S = \int d^{d+1}x\sqrt{-g}\left\{\mathcal{R}_g - \frac{1}{d-1}\sum_{i,j=1}^{n} m_i m_j g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_j - \sum_{i=1}^{n} m_i(\partial\phi_i)^2 + \right.$$
$$\left. + \sum_{i=1}^{n}\epsilon_i m_i(m_i-1)\,\exp\left(-\frac{2}{d-1}\sum_{j=1}^{n} m_j\phi_j - 2\phi_i\right)\right\}. \tag{A.4}$$

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